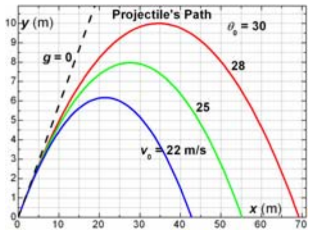
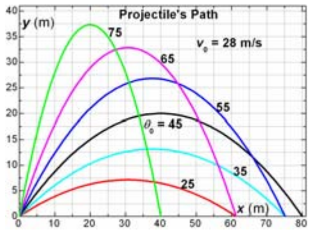
拋體運動 (補充)

例 1. 拋體軌跡, 令 $\underline{y}=y-y_0, \underline{x}=x-x_0, (x_0, y_0)$為出發點,





$$\underline{y}=\underline{x}\tan\theta_0-\frac{g\underline{x}^2}{2v_0^2\cos^2\theta_0}=-\frac{gx^2}{2v_0^2\cos^2\theta_0}(\underline{x}^2-\frac{2v_0^2\cos\theta_0\sin\theta_0}{g}\underline{x})$$

$$=-\frac{g\,x^2}{2v_0^2\cos^2\theta_0}[(\underline{x}-\frac{v_0\cos\theta_0\sin\theta_0}{g})^2-\frac{v_0^4\cos^2\theta_0\sin^2\theta_0}{g^2}]$$

$=-\alpha(\underline{x}-x_m)^2+y_m$, where (x_m, y_m) 為最高點位置,

$x_m=\dfrac{v_0^2\sin 2\theta_0}{2g}$, $\quad y_m=\dfrac{v_0^2\sin^2\theta_0}{2g}$, and $\alpha=\dfrac{g}{2v_0^2\cos^2\theta_0}$.

令 $R=x-x_0, h=y-y_0$, 解上式 x 之平方根得水平射程

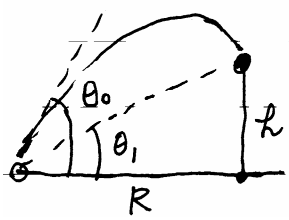
$$R=\frac{v_0^2\sin 2\theta_0}{2g}(1\pm\sqrt{1-\frac{2gh}{v_0^2\sin^2\theta_0}})\ (取+).$$

例 2. 如何投籃，才能百發百中及最省體力？

Sol. 投籃時球進網的高度 y_f 大於球出手的高度 y_i，

令 $h=y_f-y_i$, 球離手速度(v_0, θ_0)及離手位置$(0, 0)$,

球離手後的路徑為 $y=x\tan\theta_0-\dfrac{gx^2}{2v_0^2\cos^2\theta_0}$.



Case A. 給定 R, h & θ_0, $\quad v_0=?$

球如入網即 $x=R, y=h, h=R\tan\theta_0-\dfrac{gR^2}{2v_0^2\cos^2\theta_0}$,

得球離手速率 $v_0^2=\dfrac{gR^2}{2\cos^2\theta_0(R\tan\theta_0-h)}$; 由 $v_0^2>0$, 知 $\tan\theta_0>h/R\equiv\tan\theta_1$,

很合理地，球離手的角度θ_0須大於球至籃連線的仰角θ_1，才能將球射入網。

(1) R 小, v_0 小；R 大, v_0 大；(當三分球線外移時，較不易投進三分球)；

(2) h 小, v_0 小；h 大, v_0 大；(高身高者，玩籃球佔便宜)；

(3) θ_0 有一最佳化值，使 v_0 最小就可使球入網(飛車跨障礙時 $h<0$)。

Note: 令 $f(\theta)=\cos^2\theta(R\tan\theta-h)$，$f'(\theta)=2\cos\theta(-\sin)(R\tan\theta-h)+\cos^2\theta(R\sec^2\theta)=0$, or

$R+2h\cos\theta\sin-2R\sin^2\theta=0$, $R\sec^2\theta-2R\tan^2\theta+2h\tan\theta=0$, 得

$R\tan^2\theta-2h\tan\theta-R=0 \quad\Rightarrow\quad \tan\theta_{opt}=\dfrac{1}{R}(h\pm\sqrt{h^2+R^2})\ (>\tan\theta_1)$ (取+).

R (m)	h_{man} (m)	h (m)	θ_{opt} (°)	v_0 (m/s)
6.70	1.98	1.07	49.5	8.77
6.70	1.80	1.25	50.3	8.89
7.20	1.98	1.07	49.2	9.05
7.20	1.80	1.25	49.9	9.16

Note: h_{net} (籃高) = 3.05 m, R (三分球線) = 6.70 m (早期) or 7.20 m (現行).

Case B. 給定 R, h 及 v_0(**射擊打靶亦同**), **投籃角度**θ_0 = ? (*cf.* HRW-8e, Prob.4-51)

由 $h=R\tan\theta_0-\dfrac{gR^2}{2v_0^2\cos^2\theta_0}$, $\quad 2v_0^2\cos^2\theta_0(R\tan\theta_0-h)=gR^2$,

$$\frac{2v_0^2}{gR^2}(R\tan\theta_0-h)=\sec^2\theta_0=1+\tan^2\theta_0, \quad\text{or}\quad \tan^2\theta_0-\frac{2v_0^2}{gR}\tan\theta_0+1+\frac{2v_0^2h}{gR^2}=0,$$

得投籃角度 $\tan\theta_0 = \dfrac{v_0^2}{gR} \pm \sqrt{(\dfrac{v_0^2}{gR})^2 - (1+\dfrac{2v_0^2 h}{gR^2})} = \dfrac{v_0^2}{gR}[1\pm\sqrt{1-\dfrac{2g}{v_0^2}(h+\dfrac{gR^2}{2v_0^2})}\,]$ (有二解).

(1). $v_0^2 > g(h+\sqrt{h^2+R^2}\,)$才有解，即射球入網；

(2).因籃網及球有一定的寬度，當 v_0 大，$\Delta\theta_0$ 大，即較易射球入網。

例 **3.** 如何推鉛球，才能得到最好的成績？

Sol. 推鉛球時，球離手時離地高 h, 鉛球離手速度(v_0, θ_0), 離手位置$(0, h)$,

由 $v_y = v_0\sin\theta_0 - g\,t_R = 0 \;\Rightarrow\;$ 上升時間 $t_R = v_0\sin\theta_0/g$;

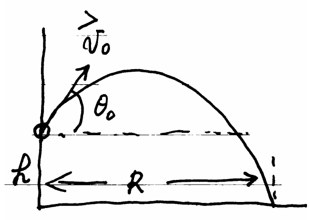
上升高度 $h_R = v_0\sin\theta_0\, t_R - \frac{1}{2}g t_R^2 = \dfrac{v_0^2\sin^2\theta_0}{2g}$,

上升時水平飛行距離 $x_R = v_{0x}\,t_R = (v_0\cos\theta_0)(v_0\sin\theta_0/g) = \dfrac{v_0^2\sin 2\theta_0}{2g}$,

下降高度 $h_F = h + h_R = \frac{1}{2}g t_F^2$ or $\frac{1}{2}g t_F^2 = h + \dfrac{v_0^2\sin^2\theta_0}{2g}$,

下降時間 $t_F = \sqrt{\dfrac{2h}{g}}\sqrt{1+\dfrac{v_0^2\sin^2\theta_0}{2gh}}$ or $t_F = \dfrac{v_0\sin\theta_0}{g}\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta_0}}$,

總飛行時間 $T = t_R + t_F = \dfrac{v_0\sin\theta_0}{g}(1+\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta_0}})$,



下降時水平飛行距離 $x_F = v_{0x}\,t_F = \dfrac{v_0^2\sin 2\theta_0}{2g}\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta_0}}$,

水平射程 $R = x_R + x_F = \dfrac{v_0^2\sin 2\theta_0}{2g}(1+\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta_0}}\,)$.

當 $\dfrac{dR}{d\theta_0} = 0 \Rightarrow$ 最佳化角度θ_{opt}, 即當 $\tan\theta_{opt} = \dfrac{1}{\sqrt{1+2gh/v_0^2}}$, R 最大.

As $h = 0$, $\theta_{opt} = 45°$; $h > 0$, $\theta_{opt} < 45°$; $h < 0$, $\theta_{opt} > 45°$.

h (m)	v_0 (m/s)	出手角度, R (m)		
1.8	14.0	40.0°, 21.65	42.6°, 21.73	45.0°, 21.66
2.0	14.0	40.0°, 21.85	42.4°, 21.91	45.0°, 21.83
2.0	14.4	40.0°, 23.00	42.5°, 23.07	45.0°, 23.00

Note: World Record = 23.12 m; 鉛球重 16 lb = 7.26 kg; R 欄中角度為 Optimal angle.

Note: 令 $f(\theta) = \sin 2\theta(1+\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta}})$, $\dfrac{df}{d\theta} = 2\cos 2\theta(1+\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta}}) + \sin 2\theta(\dfrac{1}{2})$

$\dfrac{1}{\sqrt{1+2gh/v_0^2\sin^2\theta}}\dfrac{2gh}{v_0^2}(-\dfrac{2\cos\theta}{\sin^3\theta}) = 0$, $\sqrt{1+\dfrac{2gh}{v_0^2\sin^2\theta}} + 1 + \dfrac{2gh}{v_0^2\sin^2\theta} = \dfrac{gh}{v_0^2\sin^2\theta}\dfrac{\cos\theta}{\sin\theta}$

$\tan 2\theta = \dfrac{gh}{v_0^2\sin^2\theta}\dfrac{1}{1-\tan^2\theta}$, $1+\dfrac{2gh}{v_0^2\sin^2\theta} = (\dfrac{2gh}{v_0^2\sin^2\theta}\dfrac{\tan^2\theta}{1-\tan^2\theta} - 1)^2 = 1 - \dfrac{2gh}{v_0^2\sin^2\theta}$

$\dfrac{2\tan^2\theta}{1-\tan^2\theta} + (\dfrac{2gh}{v_0^2\sin^2\theta}\dfrac{\tan^2\theta}{1-\tan^2\theta})^2$, $\dfrac{2gh}{v_0^2\sin^2\theta}\dfrac{\tan^4\theta}{(1-\tan^2\theta)^2} = \dfrac{2\tan^2\theta}{1-\tan^2\theta} + 1 = \dfrac{\sec^2\theta}{1-\tan^2\theta}$,

$\dfrac{2gh}{v_0^2} = \dfrac{1-\tan^2\theta}{\tan^2\theta}$ or $1+\dfrac{2gh}{v_0^2} = \dfrac{1}{\tan^2\theta}$, 得 $\tan\theta = \dfrac{1}{\sqrt{1+2gh/v_0^2}}$.

例 4. 棒球外野手如何判斷球的落點？(*cf.* HRW-8e, S.P.4-8, p.69)

令球離棒點為原點, $x = (v_0 \cos\theta_0)t$, $y = (v_0 \sin\theta_0 - \frac{1}{2}gt)\,t$, $R = (v_0 \cos\theta_0)\dfrac{2v_0 \sin\theta_0}{g}$,

人看球之水平距離 $R - x = (v_0 \cos\theta_0)(\dfrac{2v_0 \sin\theta_0}{g} - t) = \dfrac{2v_0 \cos\theta_0}{g}(v_0 \sin\theta_0 - \frac{1}{2}gt)$,

得選手**看球視線仰角** ϕ, $\tan\phi = \dfrac{y}{R-x} = \dfrac{g\,t}{2v_0 \cos\theta_0}$, for $t < T = \dfrac{2v_0 \sin\theta_0}{g}$,

With $v_0 = 40$ m/s and $\theta_0 = 35°$, $T = 4.68$ s ~ 4.7 s, $\tan\phi = (0.149\text{ rad})\,t$.

因此，可近似地得 $\phi \sim (0.149\text{ rad})\,t$ or $(8.57°)\,t$,亦即**看球以定速率上升**; 若站得**太近**(遠), 則看球會**急速上升**(下墜).

例 5. 棒球從離地高度 h = 1.00 m、球離開球棒的速率 v_0 = 27.0 m/s，以相對於水平面角度 $\theta_0 = 35°$朝外野飛去；設外野手離本壘距離 $d = 90.0$ m，其對球反應時間 $\Delta t = 0.50$ s，***Q***. 試問外野手之最小奔跑速率為何，才能接到球？

Sol. $v_{0x} = v_0 \cos\theta_0 = 22.90$ m/s，$v_{0y} = v_0 \sin\theta_0 = 14.31$ m/s,

$t_R = v_{0y}/g = 1.46$ s，$h_R = v_{0y}\, t_R - \frac{1}{2} g t_R^2 = 20.89 - 10.44 = 10.45$ (m),

h_F(下降高度) $= h_R + 1.00 = 11.45 = \frac{1}{2} g t_F^2$, $t_F = 1.53$ s $> t_R$.

〈1〉球剛落地時，野手接到球，球飛行時間 $T = 1.46 + 1.53 = 2.99$ (s),

$R = v_{0x}T = 68.47$，$\Delta x = d - R$，$\Delta x = 21.53$ (m),

$v_F = \Delta x/\Delta t = 21.53/(2.99–0.50) = 8.65$ (m/s).

〈2〉在空中高度 h 處，野手接到球

$T = (1.46)(2) = 2.92$ (s)，$R = v_{0x}T = 66.87$,

$\Delta x = d - R = 90 - 66.87 = 23.13$，$v_F = 23.13/(2.92–0.50) = 9.56$ (m/s).

(如發現錯誤煩請告知, jyang@mail.ntou.edu.tw, Thanks.)