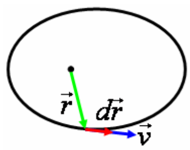
Kepler's Three Laws of Planetary Motion

Newton's law of gravitation for two particles is that every pair of particles in the universe exerts a mutual attraction with a force directed along the line joining the particles, the magnitude of the force being inversely proportional to the square of the distance between them and directly proportional to the product of their masses. $F_{12} = Gm_1m_2/r_{12}^2$ (see Fig. 1), where G is a universal constant. Let the mass of the sun be M and that of the planet be m. We shall assume that the sun is fixed at the origin of a given coordinate system (Fig. 1). The force acting on the planet due to the sun is $\vec{F} = -\dfrac{GmM}{r^3}\vec{r}$. From the second law $-\dfrac{GmM}{r^3}\vec{r} = m\dfrac{d^2\vec{r}}{dt^2} = m\dfrac{d\vec{v}}{dt}$

so that $\dfrac{d\vec{v}}{dt} = -\dfrac{GM}{r^3}\vec{r}$. (1)



Now $\dfrac{d}{dt}(\vec{r}\times\vec{v}) = \vec{r}\times\dfrac{d\vec{v}}{dt}$,

and hence $\dfrac{d}{dt}(\vec{r}\times\vec{v}) = \vec{r}\times(-\dfrac{GM}{r^3}\vec{r}) = 0$. (2)

This implies $\vec{r}\times\vec{v}$ = constant vector, (3a)

or $\vec{r}\times(d\vec{r}/dt) = \vec{h}$. (3b)

Since $|\vec{r}\times d\vec{r}|$ = twice sectoral area, we have $dA/dt = h/2$, or equal areas are swept out in equal intervals of time. This is Kepler's **second** law of planetary motion. Moreover $\vec{r}\cdot[\vec{r}\times(d\vec{r}/dt)] = \vec{r}\cdot\vec{h} = 0$ so that $\vec{r}$ remains perpendicular to the fixed vector $\vec{h}$, and the motion is planar. Now

$$\frac{d\vec{v}}{dt}\times\vec{h} = -\frac{GM}{r^3}\vec{r}\times\vec{h} = -\frac{GM}{r^3}\vec{r}\times(\vec{r}\times\vec{v}).$$

From eq. 3, and $d(\vec{v}\times\vec{h})/dt = (d\vec{v}/dt)\times\vec{h}$, so that

$$\frac{d}{dt}(\vec{v}\times\vec{h}) = -\frac{GM}{r^3}\vec{r}\times(\vec{r}\times\vec{v}). \qquad (4)$$

Now $\vec{r} = r\hat{r}$, where $\hat{r}$ is a unit vector. Hence

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r},$$

so that eq. 4 becomes

$$\frac{d}{dt}(\vec{v}\times\vec{h}) = -\frac{GM}{r^3}.\vec{r}\times(\vec{r}\times r\frac{d\hat{r}}{dt})$$

With $\hat{r}\times[\hat{r}\times(d\hat{r}/dt)] = [\hat{r}\cdot(d\hat{r}/dt)]\hat{r} - (\hat{r}\cdot\hat{r})(d\hat{r}/dt)$, we have

$$\frac{d}{dt}(\vec{v}\times\vec{h}) = GM\frac{d\hat{r}}{dt}, \qquad (5)$$

since $\hat{r}$ is a unit vector.

Integrating eq. 5, we obtain

$$\vec{v}\times\vec{h} = GM\,\hat{r} + \vec{k}$$

and
$$\vec{r}\cdot(\vec{v}\times\vec{h}) = \vec{r}\cdot(GM\,\hat{r} + \vec{k}),$$
$$(\vec{r}\times\vec{v})\cdot\vec{h} = GMr + \vec{r}\cdot\vec{k} = h^2,$$
$$h^2 = GMr + rk\cos\theta. \qquad (6)$$

Here we choose the direction of the constant vector $\vec{k}$ as the polar axis. Thus

$$r = \frac{h^2/GM}{1+(k/GM)\cos\theta}. \qquad (7)$$

This is the polar equation of a conic section. For the planets these conic sections are closed curves so that we obtain Kepler's **first** law, which states that the orbits of the planets are ellipses with the sun at one of the foci.

Let us now write the ellipse in the form

$$r = \frac{ep}{1+e\cos\theta},$$

where $e = k/GM$, $p = h^2/k$. The curve crosses the polar axis at $\theta = 0$ or π, so that the length of the major axis is

$$2a = \frac{ep}{1+e} + \frac{ep}{1-e} = \frac{2ep}{1-e^2} = \frac{2h^2}{GM(1-e^2)}.$$

For an ellipse $b^2 = a^2 - c^2 = a^2 - e^2a^2$, or $b = a(1-e^2)^{1/2}$. The area of ellipse is $A = \pi ab = \pi a(1-e^2)^{1/2}$, and since $dA/dt = h/2$, the period of one complete revolution is

$$T = \frac{A}{h/2} = \frac{2\pi a^2(1-e^2)^{1/2}}{a^{1/2}G^{1/2}M^{1/2}(1-e^2)^{1/2}} = \frac{2\pi a^{3/2}}{G^{1/2}M^{1/2}}.$$

Thus
$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}. \qquad (8)$$

This is Kepler's **third** law, which states that the squares of the periods of revolution of the plants are proportional to the cubes of the mean distances from the sun.

克普勒行星運動定律：**1.**第一(**軌道**)定律：太陽系之行星，各在以太陽為焦點之一橢圓軌道上運行。**2.**第二(**面積**)定律：由太陽連至行星之線，於相等時間內掃過相等的面積。**3.**第三(**週期**)定律：行星距太陽之平均距離 R 之立方，與行星繞太陽周期 T 之平方的比值 R^3/T^2，對各個行星皆相等。

Ref. H. Lass, *Vector and tensor analysis*, McGraw-Hill, 1975. (*NTOU981010*)